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Matrix Theory in Curved Space

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We study curved-space versions of matrix string theory taking as a definition of the theory a gauged matrix sigma model. By computing the contribution to the one-loop divergent terms in the effective action coming from the diagonal matrix elements we show that these versions of matrix theory in curved space reproduce the string equations of motion and the R^4 correction to the Hilbert–Einstein action. It is then demonstrated that the divergences due to the nondiagonal elements induce terms in the effective action that cannot be removed by appropriate counterterms. This implies that the model can only be consistent for Ricci flat manifolds with vanishing six-dimensional Euler density.

1. INTRODUCTION

Matrix string theory [2–4] is hoped to provide a nonperturbative definition of type IIA string theory. Although a full analysis of the string iteractions within the framework of matrix theory is lacking, it is believed that free strings emerge naturally from the moduli space of low-energy configurations of matrix theory. More precisely, the free string states are formed from "winding sectors" in which large numbers of eigenvalues form, via twisted boundary conditions, long string configurations composed of an order-N number of eigenvalues. The large-N limit corresponds to taking a finer and finer discretization of the light-cone string world-sheet into infinitesimal strips, and corresponds to taking the conformal limit of the theory. From this conformal field theory point of view string interactions can be argued to be described by an irrelevant, local CFT operator [4].

One of the main features of matrix string theory is the intrinsically noncommutative nature of the model. Indeed although the theory reduces to independent copies of the light-cone Green–Schwarz action of type IIA string

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theory when reduced to diagonal matrices, there is no such interpretation when the matrices are nondiagonal. For this reason it not trivial to try to extend the matrix string theory to curved space. There is a more physical obstruction which is linked to the nature of the light-cone frame. In superstring theory in 10-dimensional Minkowski space the light cone can be defined globally; in a curved background the light cone is only local. In particular it is not clear that one can define a curved version of matrix string theory directly in the light-cone gauge without missing some of the global features of string theory on a curved background.

Despite these provisos it is possible to define a natural generalization of matrix string theory using supersymmetric gauged sigma models [8–10]. These models are closely related to the curved background versions of string theory. In particular, we shall compare our results to the string equations of motion in a curved background. In string theory the classical equations of motion for the effective theory can be found either from tree level string scattering amplitudes or from the consistency conditions (conformal invariance) of the action for a noninteracting string propagating in a curved space with background fields. Similarly, it is worth exploring what can be learnt from the consistency conditions on curved-space versions of matrix string theory. We study here the conditions imposed on the curved-space versions of matrix string theory in which perturbative string theory is hoped to be recovered. We will show below that such models are only consistent for an extremely limited class of manifolds, i.e., Ricci flat manifolds with vanishing Euler class. An example is provided by the direct product $\mathbf{M} = \mathbf{S} \times \mathbf{C}$, where $\mathbf{S}$ is a hyper-Kähler surface. This is, for instance, the case of the ALE spaces [11].

We begin in Section 2 by recalling the essential elements of matrix string theory. In Section 3 we review the proposals for D-brane actions/matrix theory in curved space put forward by Douglas *et al.* [8–10]. In Section 4 we focus on the divergent part of the calculation of the effective action and show that it reduces to a simple matrix generalization of the string theory beta function calculation. Having mapped the calculation to that of the string beta function, we use known four-loop results [12] to demonstrate that the effective action can only be consistent for Ricci flat manifolds with vanishing six-dimensional Euler density. More details can be found in ref. 1.

2. $N = 8$ TWO-DIMENSIONAL SUPER YANG–MILLS AND MATRIX STRING THEORY

In this section we summarize the essential ingredients of the correspondence between the dimensional reduction of 10-dimensional super-Yang–Mills theory and type IIA string theory. The two-dimensional action reduces to

$$S = \int d\tau\, d\sigma\, \mathrm{Tr}\left[\frac{1}{2}(D_\alpha X^I)^2 + \frac{i}{2}\Theta^T\gamma^\alpha D_\alpha\Theta - \frac{1}{4}F_{\alpha\beta}^2 + \frac{1}{4g_s^2}[X^I, X^J]^2 + \frac{1}{2g_s}\Theta^T\gamma_I[X^I, \Theta]\right] \tag{1}$$

The fields are $N \times N$ Hermitian matrices. The index I runs from 1 to 8 and the 16 fermions split into the 8_s and 8_c spinorial representations of $SO(8)$. The string coupling constant of the type IIA string theory is g_s. The coordinate σ lives between 0 and 2π.

According to ref. 4, the weakly coupled string is to be obtained from the $g_s \to 0$ limit corresponding to the infrared limit of the SYM theory. In this regime the matrices commute and describe strings in the light-cone frame. The corresponding action evaluated for these configurations is the sum of N replicas of the light-cone Green–Schwarz action. In this limit the matrix coordinates can always be diagonalized using unitary transformations U

$$X^I = Ux^IU \tag{2}$$

The matrix U is defined up to an element g of the Weyl group of $U(N)$ permuting the eigenvalues,

$$U(\sigma + 2\pi) = U(\sigma)g, \qquad x^I(\sigma + 2\pi) = gx^I(\sigma)g^\dagger \tag{3}$$

The infrared regime is then identified with the two-dimensional conformal field theory described by the $N = 8$ sigma model on the target space

$$S^NR^8 = (R^8)^N/S_N \tag{4}$$

The freely propagating strings in the light-cone frame are identified in the limit $N \to \infty$ with the cycles of the eigenvalues x^I under the permutation group.

It is useful at this point to reformulate the action (1) in a way that generalizes easily to a curved background. A supersymmetric and gauge-invariant action can be written using the $d = 4$, $N = 1$ superfield formalism. The four gauge fields belong to a vector multiplet V, while the bosonic fields belong to three chiral multiplets Φ^i. The eight bosonic fields of the original action thus split into a group of six belonging to the three chiral multiplets and two obtained by dimensional reduction of the 4D gauge fields belonging to the vector multiplet. This formulation breaks the global $SO(8)$ symmetry into $SO(6) \times SO(2)$. The full $SO(8)$ symmetry is restored by going to the Wess–Zumino gauge. The Lagrangian is

$$S = \frac{1}{\alpha'}\mathrm{tr}\left(\int d^2x\, d^4\theta\, e^{gV}\Phi\, e^{-gV}\,\Phi^\dagger + \frac{1}{64g^2}\int d^2x\, d^4\theta\, W^2 + \frac{ig}{3!\sqrt{\alpha'}}\int d^2x\, d^2\theta\, \epsilon_{ijk}\Phi^i\Phi^j\Phi^k + \mathrm{cc}\right) \tag{5}$$

where $g^{-2} = \alpha' g_s^2$ is the YM coupling constant and $W_\alpha = \bar{D}^2 \epsilon^{gV} D_\alpha \epsilon^{-gV}$. The two-derivative Lagrangian of the 4D $N = 4$ SYM theory is finite; in particular the beta function vanishes.

3. CURVED SPACE ACTIONS

Candidate formulations for D-brane actions in curved space have been been proposed in refs. 8–10. For small curvatures a single D-brane is described by the Born–Infeld theory. The crucial point is that this contains a $U(1)$ gauge field which becomes non-Abelian when N D-branes coincide. In the low-energy regime this reduces to a SYM theory on the world volume of the D-branes. In curved space the D-brane action should combine the non-Abelian nature of the gauge theory and a fraction of the original 16 supersymmetries preserved by the D-brane configuration.

A set of axioms has been proposed in ref. 10 to describe the possible actions. A particularly natural set of D-brane actions in this context are those obtained from the the dimensional reduction of a 4D ($N = 1$) $U(N)$ SYM theory to $d + 1$ dimensions [8]. The curved background is a 3D complex Kähler manifold whose metric depends on a Kähler potential K. The vector superfields contain $(3 - d)$ real, flat coordinates. Notice that the splitting of the background manifold implies that the original $SO(8)$ global symmetry is reduced to $SO(3 - d)$. The case $d = 1$ corresponds to the matrix string theory, while $d = 0$ is a curved version of the matrix model for M-theory.

In a setting adapted to our purposes the axioms amount to the following two requirements for the D-brane action defined on a 3-dimensional Kähler manifold $\mathcal{M}$.

(a) The classical moduli space, determined by the vanishing of the D and F terms of the SYM theory, is the symmetric product $\mathcal{M}^N/S_N$.

(b) The action is a single trace.

These axioms imply that the action in curved space reads

$$S = \frac{1}{\alpha'} \mathrm{tr} \Bigg[\int d^{d+1}x \, d^4\theta \, K(e^{gV} \Phi \epsilon^{-gV}, \Phi^\dagger) + \left(\int d^{d+1}x \, d^4\theta \, W(\Phi) + \frac{1}{64 g^2} \int d^{d+1}x \, d^2\theta \, W^\alpha W_\alpha + \mathrm{cc} \right) \Bigg] \tag{6}$$

The analysis of axiom (a) leads to the following form for the superpotential:

$$W = \epsilon_{ijk} a^i(\Phi)[\Phi^j, \Phi^k] \tag{7}$$

where $a^i(\Phi)$ is a holomorphic vector field in the adjoint representation of the gauge group. We use the most general Kähler potential allowed by supersymmetry and gauge invariance. We will use the fact that there exists around each point of the moduli space a set of normal Kähler coordinates. These coordinates are such that locally

$$K(z, \bar{z}) = z\bar{z} + \Sigma \frac{1}{L_R^{n-2}} K_{I_1 \cdot\cdot I_p \bar{I}_{p+1} \cdot\cdot \bar{I}_n} z^{I_1} \cdot\cdot z^{I_p} \bar{z}^{I_{p+1}} \cdot\cdot \bar{z}^{n} \tag{8}$$

The existence of this expansion is guaranteed up to an analytic change of coordinates on the curved manifold. By definition the $K_{I_1 \cdot\cdot \bar{I}_p I_{p+1} \cdot\cdot \bar{I}_n}$ are symmetric with respect to arbitrary reorderings of the holomorphic indices and arbitrary reorderings of the antiholomorphic indices. Finally, since we are dealing with matrices, there is a question of ordering in the Kähler potential. The most natural ansatz is to assume that all terms in the Kähler potential are symmetrized products of matrices, but there could be more general orderings. The fourth-order term, for example, can be written as

$$K_{IK\bar{J}\bar{L}} \, [\delta \Phi^I \Phi^K \Phi^{\bar{J}} \Phi^{\bar{L}} + \tau \, \Phi^I \Phi^{\bar{J}} \Phi^K \Phi^{\bar{L}}] \tag{9}$$

where δ and $\tau = 1 - \delta$ are constants. It is also possible for the Kähler potential to contain terms proportional to commutators of matrices since these vanish for the classical moduli space (diagonal matrices). In fact it was found in ref. 10 that imposing the axioms stated above constrains the fourth-order term to be the totally symmetrized product ($\delta = 2/3$, $\tau = 1/3$) with no additional terms corresponding to commutators.

Let us first use a very naive argument to justify the link between the matrix string theory on a curved background and the type IIA string theory in curved space. Substituting the diagonal matrices describing the moduli space $\mathcal{M}$ in the action leads to a sum of N copies of the $U(1)$ gauged sigma model in two dimensions. The gauge part of the action describing the flat component of the background manifold decouples and one is left with N copies of the sigma model defined by the background curved manifold

$$\frac{1}{\alpha'} \int d^2x \, d^4\theta \, K(\Phi, \Phi) \tag{10}$$

where Φ represents one of the N components. The analysis of this action reveals that there are two-dimensional UV divergences. These logarithmic divergences can be canceled up to three-loop order by imposing that the Ricci tensor vanishes

$$R_{I\bar{J}} = 0 \tag{11}$$

This is the usual Einstein equation as deduced from the conformal invariance of string theory. At four-loop order this is no longer true; the beta function is nonzero for Ricci-flat manifolds. The divergence is proportional to

$$R_{hkmn}R_{rs}^{h\ n}(R^{ksrm} + R^{kmrs}) \tag{12}$$

This is equivalent to the result obtained from the calculation of the four-graviton scattering for type IIA theory. This leads to a correction of the effective 10D supergravity action and the familiar R^4 term.

It seems therefore that a naive application of matrix theory in curved space leads to the correct identification of the string equations. This is misleading, as a detailed analysis expounded in the following will show.

4. THE EFFECTIVE ACTION IN A CURVED BACKGROUND

In the previous section we have defined the curved background version of the matrix string theory. This involves an explicit splitting between the six curved coordinates represented by a nonlinear sigma model coupled to $SU(N)$ YM fields and the two coordinates obtained by dimensional reduction of the four-dimensional YM gauge fields. We are interested in the equivalence between this theory and string theory in a curved background. In particular we have seen that a naive calculation of the effective action for diagonal configurations leads to the string equations. In this section we reexamine this issue by properly integrating over the background fluctuations to arrive at an effective action for the diagonal configuration. We will focus solely on the divergent contributions to the Kähler potential. We will show that the resulting effective action can only be consistent for a very limited class of manifolds.

4.1. Superfield Reduction

Separating the chiral superfields and the vector superfields into diagonal and off-diagonal parts, the effective action for the diagonal fields is obtained by integrating over the off-diagonal elements ϕ and v and the fluctuations of the diagonal parts ϕ_d and v_d. The resulting effective action possesses a modified Kähler potential K_R in such a way that

$$S_{\text{eff}} = \frac{1}{\alpha'}\int d^2x\, d^4\theta\, K_R(\Phi_d, \Phi_d) \tag{13}$$

The superpotential is not renormalized and vanishes for diagonal configurations. The renormalized Kähler potential is obtained after removing the UV divergences leading to poles in $1/\epsilon$ when using dimensional regularization. These poles correspond to the logarithmic divergences of the sigma models in two dimensions. One can proceed systematically to find the divergence for a general loop diagram. If one has L loops with P propagators of any type, C of which are chiral–antichiral propagators, we have the degree of divergence

$$\text{div} = 2L - 2P + 2C - 2L = -2(P - C) \tag{14}$$

We thus see that for divergent diagrams $P = C$. We are thus left with examining the divergences due to chiral diagrams with no superpotential insertions and no gauge fields.

4.2. Chiral Diagrams

As stated above, it is known that in string theory the one-, two-, or three-loop divergent contributions disappear for Ricci flat manifolds, while at four loops there is a correction that only disappears for manifolds with a vanishing six-dimensional Euler density. The divergences lead to the famous R^4 term being added to the low-energy effective action for the massles modes of the string. In other words, Ricci flatness is a low-order approximation corrected by terms of higher order in α'.

For the curved-space versions of matrix theory, however, there are two types of chiral diagrams. First there are those coming from the expansion of the Kähler potential in terms of the diagonal fluctuations only. This is nothing but N copies of the two-dimensional sigma model with values in a three-dimensional complex Kählerian manifold. Second there are diagrams involving the off-diagonal part of ϕ. These will lead to divergent terms involving one or more diagonal elements, i.e., to terms consisting of products of traces. Since these are not included in the original action, they have to be set to zero. In other words, we find that each loop order has to be individually set to zero. This is a more stringent restriction than in string theory.

Retaining only at each loop order the contribution due to the diagonal matrices is a simple generalization of the string-theory beta-function result (we study this question below). We see that, in particular, the four-loop term has to be set to zero. This implies that the curved manifold must be Ricci flat with a vanishing Euler class. This is, for instance, the case of products $\mathbf{M} \times \mathbf{C}$, where $\mathbf{M}$ is hyper-Kähler. In particular, the ALE spaces are good candidates for a description of matrix string theory in curved space.

This result thus restricts quite severely the range of applicability of matrix sigma models as descriptions of matrix theory in curved space. Up to now we have only considered the terms due to the diagonal matrices. This is not sufficient to guarantee the finiteness of the model. We now turn to off-diagonal contributions. We will only examine them at the one-loop order.

4.3. One-Loop Contribution

It is not immediately obvious that Ricci flat metrics lead to vanishing one-, two-, and three-loop contributions for the matrix sigma model. Indeed the contribution of the off-diagonal matrices needs to be carefully examined. It is important to measure their relevance at least for the first nontrivial term. Failure of the cancellation process at this level would almost certainly lead to the conclusion that matrix theory can only be consistent for flat space. The first nontrivial test involves the sixth-order term in the expansion of the Kähler potential. We show that, by a particular choice of ordering and the addition of a particular commutator term (that vanishes on the classical moduli space), this contribution will dissapear for Ricci flat metrics. The condition to be satisfied for this to be the case is identical to one of the mass conditions deduced in ref. 10.

The perturbative expansion at one-loop level is sketched in Fig. 1. Each vertex corresponds to a term in the expansion of the Kähler potential. The external lines correspond to the number of background fields Φ. The first line thus corresponds to the Ricci tensor $R_{I\bar{J}} = \delta^{K\bar{K}} K_{IK\bar{J}\bar{K}}$ evaluated at the special point about which we have chosen the normal coordinates.[2] The second line corresponds to a correction of order Φ^3, etc. Saying that the metric is Ricci flat at the point $\Phi = 0$ amounts to having the first term equal to zero. Saying that it is Ricci flat everywhere implies that every line (the coefficient for each power of Φ) is zero.

The one-loop contribution reads

$$\delta K^{1L} = \frac{1}{\epsilon} \sum_{ij} \ln(\det(g_{ij})) \qquad \text{with} \quad (g_{I\bar{J}})_{ij} = \frac{\partial^2}{\partial \Phi^I_{ij}\, \partial \Phi^{\bar{J}}_{ji}} K(\Phi, \Phi) \tag{15}$$

where the determinant is taken over the indices I and $\bar{J}$. As discussed in

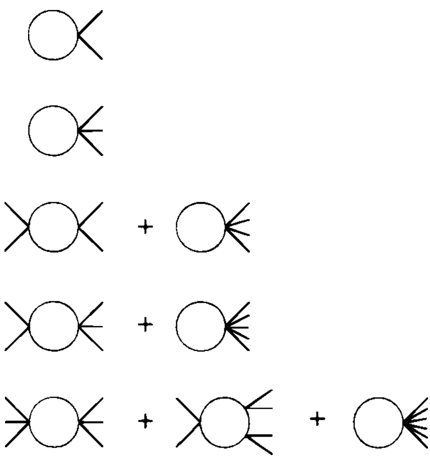

Fig. 1. Loop expansion.

[2]The series starts at order Φ^2 since this is the first relevant contribution inside the full superspace integral.

Section 4.2, this contribution has to be set to zero. Equivalently this leads to the condition that for each i, j

$$\det(g_{ij}) = 1 \tag{16}$$

This is precisely one of the mass conditions deduced in ref. 10.

This condition first becomes nontrivial for the third line of Fig. 1, which represents the sum of a term coming from all possible connected contractions of the two fourth-order terms in the Kähler potential and the contraction of single sixth-order term with itself.

In ref. 10 the fourth-order term was found to be the totally symmetrized product. The sixth-order term is more intricate,

$$\begin{aligned} K_6 = {} & K_{IKM\bar{J}\bar{L}\bar{N}}\, IKM\bar{J}\bar{L}\bar{N} \\ & -\frac{4}{3}\,\delta^{P\bar{P}}\, K_{IM\bar{J}\bar{P}}K_{KP\bar{L}\bar{N}}(MIK\bar{N}\bar{J}\bar{L} - MI\bar{J}\bar{N}K\bar{L} \\ & + \bar{N}IKM\bar{J}\bar{L} - \bar{N}I\bar{J}MK\bar{L}) \end{aligned} \tag{17}$$

where for compactness we denote the chiral fields solely by their complex indices, i.e., $I_i = \Phi_i^I$, $\bar{J}_i = \bar{\Phi}_i^J$, etc. The only other possible terms that could be added to this are terms which disappear under the contractions being considered above, i.e., terms which are zero when there are less than three nondiagonal matrices. Such terms can only be constructed from the product of three commutators. Presumably the difference between the result (17) and the complicated form presented in ref. 10 amounts to the addition of such terms.

Imposing the fact that the determinant of the metric is unity to all orders is a difficult task which has not been performed. We believe that this should be feasible order by order in perturbation theory.

5. CONCLUSIONS

We have shown that the one-loop calculation for the effective action for matrix string theory in a curved space has divergences corresponding to nonlocal terms connecting together two or more diagonal elements. These terms arise from simple matrix generalizations of the string-theory beta-function calculation. They correspond to powers of traces and, since the original action is postulated to contain a single trace, cannot be renormalized into a redefinition of the Kähler potential. This imposes that each order in the loop expansion has to be set to zero. Retaining only the diagonal contributions to the divergences leads to topological restrictions on the type of background spaces; they are Ricci flat with a vanishing Euler density. There are also constraints due to the off-diagonal terms; the metric has to be unimodular. At lowest nontrivial order it is possible to find particular matrix orderings and commutator terms that satisfy this condition.

The analysis of this article did not depend on the size of the matrices and it is hard to see any hidden subtleties in the taking of the large-N limit that might change the analysis for infinite N.[3]

It is also not at all obvious how to modify the gauged matrix sigma models to have a more general applicability. The addition by hand of powers of traces to cancel the divergences would be ad hoc and it is not clear how the inclusion of higher derivative terms could improve the problem. It seems likely that there is something more fundamental missing from the description. Certainly one is all too aware of the lack of a basic principle to guide us and the lack of a solid set of fundamental building blocks from which to construct actions. Perhaps this is another sign [17] that matrix variables are insufficient to describe curved space, even for infinite N.

REFERENCES

[1] Ph. Brax and T. Wynter, hep-th/9806176.
[2] L. Motl, hep-th/9701025.
[3] T. Banks and N. Seiberg, *Nucl. Phys. B* **497**, 41–55 (1997); hep-th/9702187.
[4] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys. B* **500**, 43–61 (1997); hep-th/9703030.
[5] T. Wynter, *Phys. Lett. B* **415**, 349–357 (1997); hep-th/9709029.
[6] S. B. Giddings, F. Hacquebord, and H. Verlinde, hep-th/9804121.
[7] G. Bonelli, L. Bonora, and F. Nesti, hep-th/9805071.
[8] M. R. Douglas, Lectures at Strings '97 hep-th/9707228.
[9] M. R. Douglas, *Adv. Theor. Math. Phys.* **1**, 198–209 (1998); hep-th/9703056.
[10] M. R. Douglas. A. Kato, and H. Ooguri, *Adv. Theor. Math. Phys.* **1**, 237–258 (1998); hep-th/9708012.
[11] M. R. Douglas, and G. Moore, hep-th/9603167.
[12] M. T. Grisaru, A. E. M. Van de Ven, and D. Zanon, *Nucl. Phys. B* **277**, 388–408, 409–428 (1986).
[13] M. Grisaru, M. Rocek, and W. Siegel, *Nucl. Phys. B* **159**, 429–450 (1979).
[14] S. Gates, M. Grisaru, M. Rocek, and W. Siegel, *One Thousand and One Lessons in Supersymmetry* (Benjamin-Cummings, 1983).
[15] P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, Singapore, 1986).
[16] T. Wynter, *Phys. Lett. B* **439**, 37 (1998).
[17] M. R. Douglas, H. Ooguri, and S. Shenker, *Phys. Lett. B* **402**, 36–42 (1997); hep-th/9702203.

[3] Some subtleties in the large N limit have been pointed out in ref. 16, but they are infrared effects, not ultraviolet.